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AN EXACT SOLUTION TO THE TWO POINT DC
(OPERATING-CHARACTERISTIC) CURVE DE. (U) HAROLD LEONARD
STUART SCHOOL OF MANAGEMENT AND FINANCE CHICAG.

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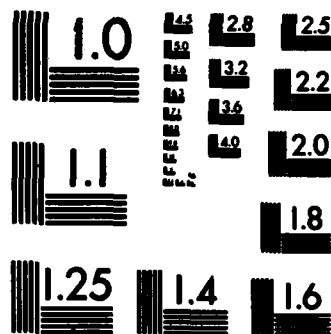
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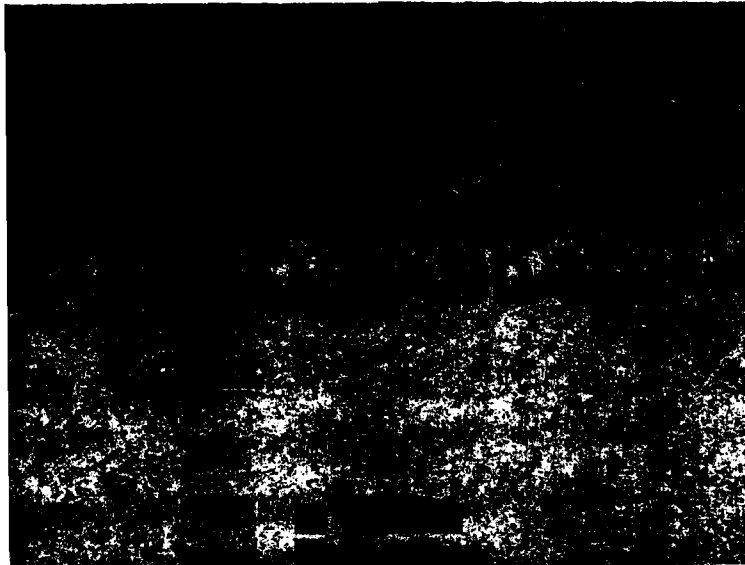


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AN EXACT SOLUTION TO THE TWO
POINT OC CURVE DETERMINATION

James F. Hall*
Thomas W. Knowles **
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November, 1983



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ABSTRACT

The determination of an operating-characteristic (OC) curve which passes through two arbitrarily selected points is a well known problem in statistical quality control. Ordinarily this problem cannot be solved exactly as the solution sampling plan must be defined by a non-negative integer couple (n, c) .

In this paper a randomization process is defined for the acceptance and rejection of all lots. Within this randomization process the two point OC definition problem is cast as a linear program. Employing the traditional solution and the LP solution, a procedure is devised for an exact solution to the problem.

KEY WORDS

OC Curve

Sampling Plans

Randomized Acceptance

Linear Programming



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1. INTRODUCTION

A well known problem in lot by lot attribute sampling is the determination of a single sampling plan whose OC curve passes through two predetermined points. Usually the points selected are the process averages and the corresponding α and β values representing levels of the producer's and the consumer's risks. Ordinarily this problem cannot be solved exactly as the number couple, (n, c) , defining the sampling plan must consist of integers. An easily acquired approximate solution to the problem and tables for its implementation are given in Grubbs [3] and Duncan [2].

A reformulation of the problem, which is soluble, requires the determination of (n, c) such that $P_a(p_1) \geq 1 - \alpha$ and $P_a(p_2) \leq \beta$ for the smallest possible c . The function $P_a(p)$ defines the OC curve of the plan and p_1 and p_2 are values of the process average. It is assumed that, $p_1 < p_2$ and $1 - \alpha > \beta$. Hald [4,5] offers a lucid survey of both exact and approximate solutions to this problem. Stephens [7] derives a closed form solution to the exact problem by employing a normal curve approximation to the binomial distribution given by Börges [1]. The solution of Stephens is extended by Jaech [6].

The solutions referred to represent a compromise since ideally we seek a sampling plan whose OC curve satisfies the two points exactly.

In this paper we associate pre-determined probabilities with each outcome of the sampling process. The sampled lot is then accepted or rejected accordingly as a random event occurs or fails to occur based on these probabilities. We refer to this procedure as the randomized acceptance plan. Within this randomization procedure we can determine the probabilities which cause the OC curve of the plan to pass through any two points exactly.

2. RANDOMIZED ACCEPTANCE SAMPLING PLAN

We propose to develop a randomized acceptance sampling plan in which a given lot is accepted or rejected accordingly as a random event occurs or fails to occur, respectively. To this end, consider the set of numbers $\{\gamma_0, \gamma_1, \dots, \gamma_n\}$, where $0 \leq \gamma_i \leq \gamma_{i+1} \leq 1$ for $i = 0, 1, \dots, n-1$.

Subsequent to extracting a random sample of size n from the lot, the randomized plan is executed by observing x , the number of defectives in the sample, and accepting the lot with probability $1-\gamma_x$ or rejecting the lot with probability γ_x . Just as the traditional OC curve defines the average fraction of accepted lots for a given level of the process average, the mathematical expectation of $(1-\gamma_x)$ defines the average fraction of lots which are accepted under the randomized plan for a given level of the process average. Therefore, while P_a defines the operating characteristics of the traditional plan, the value $E(1-\gamma_x)$ defines the operating characteristics of the randomized sampling plan.

The traditional plan is defined by (n, c) and the randomized sampling plan is defined by $\{\gamma_0, \gamma_1, \dots, \gamma_n\}$.

In the notation of the randomized plan the traditional plan becomes

$$\gamma_x = \begin{cases} 0, & \text{for } x = 0, 1, \dots, c \\ 1, & \text{for } x = c+1, \dots, n. \end{cases}$$

Obviously, under the randomized plan as defined above,

$$\begin{aligned} E(1-\gamma_x) &= (1-\gamma_0) b_0 + \dots + (1-\gamma_n) b_n \\ &= b_0 + b_1 + \dots + b_c \\ &= P_a, \end{aligned}$$

where b_x , $x = 0, 1, \dots, n$, is the probability mass function of x , the number of defectives in the sample of size n . Usually b_x will be defined by the binominal or Poisson model.

3. THE TWO POINT OC PROBLEM AS A LINEAR PROGRAM

Within the randomized sampling plan concept we define the two point OC curve problem as follows. Having selected the points $(p_1, 1-\alpha)$ and (p_2, β) , $p_1 < p_2$, $1-\alpha > \beta$, find a set of numbers $\{\gamma_0, \gamma_1, \dots, \gamma_n\}$ such that,

$$\begin{aligned} E_1(1-\gamma_x) &= 1-\alpha \\ E_2(1-\gamma_x) &= \beta \end{aligned} \quad (1)$$

subject to $0 \leq \gamma_i \leq \gamma_{i+1} \leq 1$, $i = 0, 1, \dots, n-1$. The symbols E_1 and E_2 denote mathematical expectations calculated at p_1 and p_2 . An approximate statement of (1) is obtained by casting the problem as a linear program, thus,

$$\min S_1 + V_1 + S_2 + V_2 \quad (2)$$

subject to:

$$\left. \begin{aligned} \gamma_0 b_{01} \quad \gamma_1 b_{11} + \dots + \gamma_n b_{n1} + S_1 - V_1 &= \alpha \\ \gamma_0 b_{02} \quad \gamma_1 b_{12} + \dots + \gamma_n b_{n2} - S_2 + V_2 &= 1 - \beta \end{aligned} \right\} \quad (3)$$

$$\left. \begin{aligned} \gamma_0 - \gamma_1 &\leq 0 \\ \gamma_1 - \gamma_2 &\leq 0 \\ &\vdots \\ \gamma_{n-1} - \gamma_n &\leq 0 \\ \gamma_0 &= 0 \\ \gamma_n &= 1 \end{aligned} \right\} \quad (4)$$

$$\gamma_i, S_1, V_1, S_2, V_2 \geq 0. \quad (5)$$

$i = 0, 1, \dots, n$,

where

$$b_{xj} = \frac{e^{-z_j} z_j^x}{x!}, \quad x=0, 1, \dots, \infty$$

and $z_j = np_j$, $j=1, 2$.

We have chosen to model b_{xj} as Poisson although the binomial, the hypergeometric or any relevant probability mass function on x would be appropriate.

The variables S_1 and S_2 are non-negative error variables measuring the extent to which the OC curve of the randomized acceptance plan exceeds $1-\alpha$ at p_1 and lies below β at p_2 , respectively. Analogously, the variables V_1 and V_2 are error variables which define the distances the OC curve of the randomized plan lies below $1-\alpha$ at p_1 and above β at p_2 . It is apparent that S_1 and V_1 cannot be positive together. A similar statement is implied for S_2 and V_2 .

The constraints defined in (3) accommodate OC curve departures from the equality defined in (1) where the constraints defined in (4) represent a general characterization of the usual properties of the operating characteristic curve. In the language of linear programming, the constraints of (5) are the usual LP non-negativity restrictions. Setting $\gamma_0 = 0$ and $\gamma_n = 1$ assure the OC curves of the randomized plan to pass through (0,1) and (1,0) under the binomial and hypergeometric model assumptions. All these constraints are compatible with the properties of the traditional OC curve and are intuitively acceptable.

Within the structure of this problem, our goal is more extensive than the apparent minimization of the objective function as stated in (2). In addition to seeking the set $\{\gamma_0, \gamma_1, \dots, \gamma_n\}$ for which $S_1 + V_1 + S_2 + V_2$ is least we also desire the smallest value of n for which the sum of the errors is equal to zero. Surely all values of n ($n > 0$) produce basic feasible solutions to the linear programming problem as stated, so we turn to the problem of finding $\{\gamma_0, \gamma_1, \dots, \gamma_n\}$ for minimum n such that $S_1 + V_1 + S_2 + V_2$ is equal to zero.

4. DETERMINATION OF INITIAL n

From the information in Grubbs' table we easily obtain four traditional sampling plans, all approximate solutions to the two-point OC curve problem.

Of these four sampling plans, the OC curve of one passes through $(p_1, 1 - \alpha)$ and lies above β at p_2 . Another plan has an OC curve which passes through (p_2, β) and lies below $1 - \alpha$ at p_1 . We designate these OC curves as less discriminating than the OC curve of the solution plan we seek. On the other hand, of the two remaining approximate solution sampling plans, one has an OC curve which passes through $(p_1, 1 - \alpha)$ and lies below β at p_2 , whereas the other passes through (p_2, β) and lies above $1 - \alpha$ at p_1 . We designate the OC curves of these two sampling plans as more discriminating than the OC curve of the solution sampling plan we seek. The two less discriminating OC curves are characterized by sampling plans which have smaller sample sizes than those of the more discriminating OC curves. We feel intuitively that the randomized plan we seek is defined by a sample size intermediate to the sample sizes of the two groups of OC curves just discussed. Therefore, as a starting point or an initial sample size, we select the larger of the two sample sizes of the less discriminating OC curves. From this initial n we will increment upward by trial and error until we reach the least n for which $S_1 + V_1 + S_2 + V_2$ is equal to zero.

5. THE OPTIMUM SOLUTION: AN EXAMPLE

In this section we shall illustrate the solution procedure by an example. The problem we have selected is presented by Duncan (1974) and is defined as follows. We seek a sampling plan which will reject submitted material which is 1% defective 5% of the time, while accepting submitted material which is 8% defective only 10% of the time. In other words, we seek a sampling plan whose OC curve passes through the two points, $(0.01, 0.95)$ and $(0.08, 0.10)$. Employing Grubbs' table [3], Duncan [2]

defines four traditional sampling plans which approximate the desired plan. The characteristics of these four plans are listed in Table 1, together with their AOQL values.

Table 1. Approximate Solution Sampling Plans Passing Through (0.01, 0.95) and (0.08, 0.10)

plan no.	n	c	α	β	AOQL*
1	36	1	0.05	0.23	0.022
2	49	1	0.08	0.10	0.016
3	67	2	0.03	0.10	0.019
4	82	2	0.05	0.04	0.014

* Based on lot size of 1000

It is apparent from our previous discussion of section 4 that the first two plans of Table 1 are less discriminating than the plan we seek, while plans 3 and 4 are more discriminating than the desired plan. This means we intuitively seek an optimum solution with a value of n intermediate to 49 and 67.

We initiate a solution to the LP problem defined by (2), (3), (4) and (5) with $\alpha = 0.05$, $\beta = 0.10$, $p_1 = 0.01$, $p_2 = 0.08$ and an initial value of n of 49. LP solutions beyond $n = 49$ are executed by increasing n by one over the previous solution. These solutions are continued until the objective function equals zero. Table 2 contains solution data on sample size, number of simplex iterations for each solution and the values of each error variable at each LP solution.

Table 2. LP Results for the Randomized Plan
($\alpha = 0.05$, $\beta = 0.10$, $p_1 = 0.01$, $p_2 = 0.08$)

Solution No.	Sample Size n	No. of Simplex iterations	S_1	V_1	S_2	V_2
1	49	53	0.0000	0.0361	0.0000	0.0000
2	50	54		0.0359		
3	51	55		0.0355		
4	52	56		0.0348		
5	53	57		0.0338		
6	54	58		0.0326		
7	55	59		0.0310		
8	56	60		0.0291		
9	57	61		0.0268		
10	58	62		0.0242		
11	59	63		0.0211		0.0000
12	60	64		0.0000		0.0167
13	61	65				0.0120
14	62	66				0.0073
15	63	68				0.0028
16	64	68	0.0000	0.0000	0.0000	0.0000

The values of the γ 's and n which define the optimum randomized plan are:

$$\gamma_0 = 0.000$$

$$\gamma_1 = 0.007$$

$$\gamma_2 = 0.189$$

$$\gamma_3 = 1.000$$

$$\vdots$$

$$\gamma_{64} = 1.000$$

$$n = 64$$

(6)

It is apparent that sixteen LP solutions were required to arrive at the value of n which reduced each of the error variables to zero. This number was arrived at by incrementing upward from $n = 49$. It would have been more economical in practice to have begun with $n = 67$ and incremented downward to the optimum n at 64, requiring only four LP solutions to arrive at the optimum. However, the time required to execute a single LP solution is relatively small. The LP computation will be discussed in section 7.

The randomized plan just derived is executed by observing x , the number of defectives in the sample of size 64, and accepting the lot if $x = 0$. If $x = 1$, a biased coin which falls heads with probability 0.007 is tossed. If the coin falls heads the lot is rejected, otherwise it is accepted. If $x = 2$, a biased coin which falls heads with probability 0.189 is tossed. If the coin falls heads, the lot is rejected, otherwise it is accepted. For all values of x equal to or greater than three, the lot is rejected. The OC curve for the randomized plan is presented in Figure 1. We are assured that the OC curve of this plan passes through the points (0.01, 0.95) and (0.08, 0.10), exactly.

The similarity between the randomized plan and the traditional plan is apparent.

6. THE AOQL FOR RANDOMIZED SAMPLING PLANS UNDER RECTIFICATION

Under total rectification of rejected lots, the AOQ of the randomized sampling plan is defined as,

$$AOQ = \frac{N-n}{N} p \left[(1-\gamma_0)b_0 + (1-\gamma_1)b_1 + \dots + (1-\gamma_n)b_n \right] . \quad (7)$$

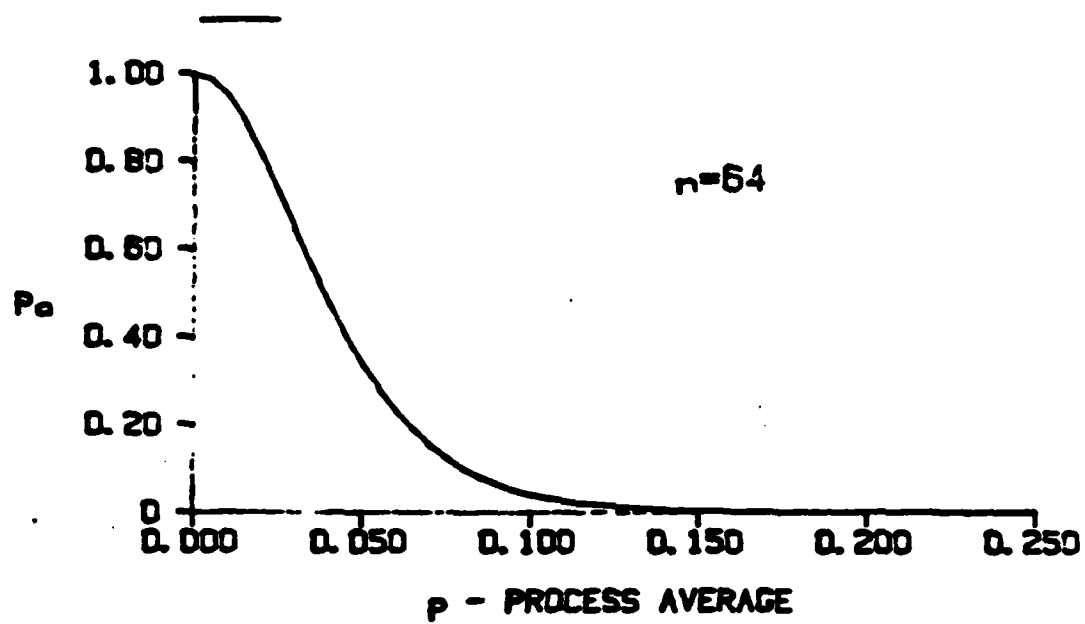


Figure 1. OC Curve - Randomized Plan

where

$0 \leq \gamma_i \leq \gamma_{i+1} \leq 1$, $i=0, 1, 2, \dots, n-1$ and n is the sample size from a lot size of N . Employing the Poisson probability distribution defined as,

$$b_x = \frac{e^{-z} z^x}{x!}, \quad x = 0, 1, 2, \dots, \dots$$

where $z = np$, gives

$$\begin{aligned} \text{AOQ} = \frac{N-n}{nN} e^{-z} & [(1-\gamma_0)z + (1-\gamma_1)z^2 + (1-\gamma_2)\frac{z^3}{2!} + \dots + \\ & + (1-\gamma_n)\frac{z^{n+1}}{n!}]. \end{aligned} \quad (8)$$

Setting

$$\begin{aligned} \frac{d(\text{AOQ})}{dz} = 0 \quad \text{from (8), and collecting like powers of } \frac{z^i}{i!} \text{ gives,} \\ 0 = (1-\gamma_0) + (1+\gamma_0 - 2\gamma_1)z + (1+2\gamma_1 - 3\gamma_2)\frac{z^2}{2!} + \dots \\ + [1+n\gamma_{n-1} - (n+1)\gamma_n]\frac{z^n}{n!} - \frac{(1-\gamma_n)z^{n+1}}{n!}, \end{aligned} \quad (9)$$

which is a necessary condition for a maximum AOQ or AOQL.

Substituting the values from (6), which define the optimum randomized plan, into (9) and solving for z gives, $z^* = 2.182$. Substituting z^* for z in the expression for the AOQ defined in (8) gives the AOQL of the randomized plan as 0.0184. It is apparent that the AOQL of the randomized plan is compatible with the AOQL values of the approximate solution plans which are presented in Table 1. The AOQ curve of the randomized plan is shown in Figure 2.

Under randomized plans involving rectification, rejected lots are screened 100% just as they are under traditional plans.

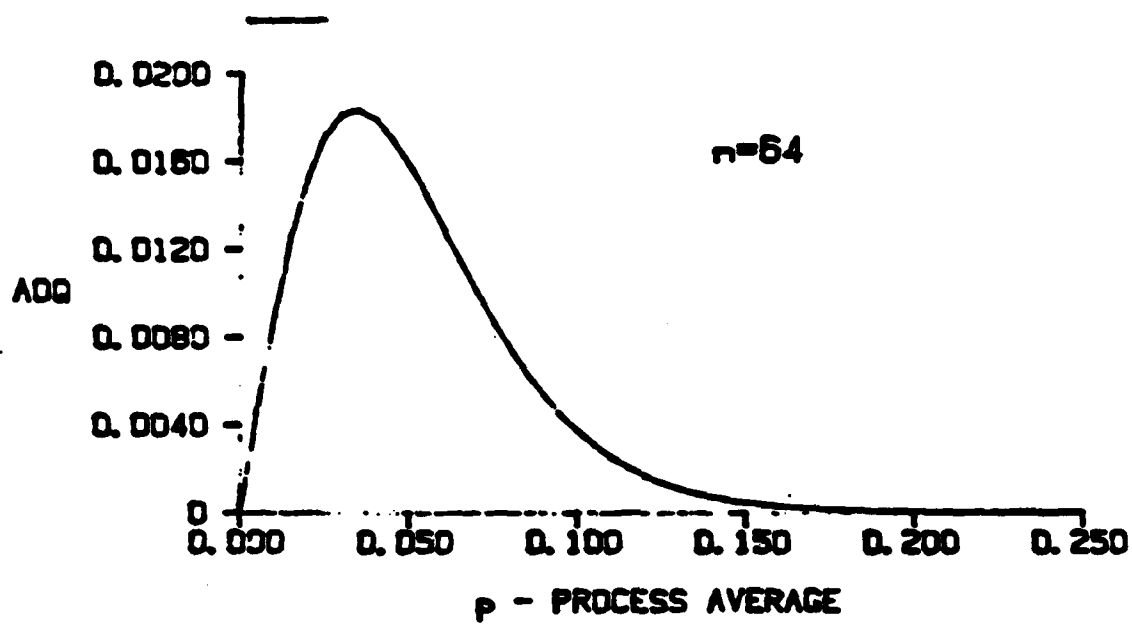


Figure 2. ADD Curve - Randomized Plan

7. COMPUTING THE LP SOLUTIONS

The generation of the linear programming model for computing purposes was done by a matrix generator subroutine written for LINDO (Linear Interactive Discrete Optimizer), a generally available linear programming computer package. The subroutine requests the user to specify n , p_1 , α , p_2 and δ , and from these input data automatically generates the desired linear programming model. With the use of this system, generating the randomized sampling plans for various values of n is done interactively, thus model input and optimum solution require only a few minutes. This subroutine is available from the authors.

8. CONCLUDING REMARKS

We have introduced the notion of a randomized sampling plan, a general concept for a single sampling inspection procedure by attributes. Within the randomized sampling concept we have cast the two point OC curve problem as a linear program, resulting in an exact solution to the problem. In addition, we have demonstrated how to determine the AOQL for a specifically defined randomized sampling plan. Thus, we have demonstrated the adaptability of the randomized plan to sampling inspection with rectification.

In addition to solving the two point OC curve problems exactly, the authors feel that the generality and ease of application of the randomized plan, together with the power of the LP solution technique provide a useful and ready means for investigating many relevant problems in sampling plan definition.

9. ACKNOWLEDGEMENT

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